DESIGN COMPUTING + COGNITION '04

WORKSHOP 2

Curves and surfaces in generative design

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SHAPE ALGEBRAS FOR CURVES AND SURFACES

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A unified foundation for shape arithmetic in any shape algebra, including curves and surfaces, exists (Krishnamurti and Stouffs, 2004). A shape algebra is defined in terms of the least element 0 (the empty shape), the *part* or *subshape* relation, $\leq$, and the operations of *sum* ($x + y$ is the least upper bound), *product* ($x \cdot y$ is the greatest lower bound), *difference* ($x - y$ is the least shape z that solves the equation $x = z + x \cdot y$) and *symmetric difference* ($x \oplus y = (x - y) + (y - x) = x + y - x \cdot y$). A shape is a subshape of another shape if every spatial part of the first shape is part of the second (see Stiny, 1986, who has written extensively on this notion).

We consider shape algebras in which shapes are collections of spatial elements of limited but nonzero measure. Each spatial element is specified by a *type* and a *form*. Both the type and the form are themselves described by a shape. The type of a spatial element is described by a shape in which the element is embedded, that is, the *carrier*, $c(x)$, of the spatial element. We say that a shape is of the same *type* as its elements. For example, line shapes are collections of finite lines; curve shapes are collections of finite curves; surface shapes are collections of finite surfaces. A shape from a cartesian product of shapes has the 'compound' type of its constituent shape types. The type of an element or shape acts as a filter for distinguishing categories of shapes.

The form of a spatial element is also described by a shape, but of a different type. For example, the form of a finite line is defined by its pair of (end) points, the form of a finite curve is also defined by its pair of (end) points; the form of a finite surface is defined by a curve (and/or line) shape and so on. The shape that describes the form of an element, i.e., its *boundary*, $b(x)$, is necessarily of a different type than the shape itself, typically, that of a shape of a lower dimensional type. A spatial element can always be reduced to one or more segments with 'minimal' boundary with respect to the spatial element: A spatial element or shape is a *segment* if it has no nonempty proper subshape the boundary of which is a subshape of the boundary of the segment (x is a segment if and only if there is no shape $y \neq 0$, $y \neq x$, such that $y \leq x$ and $b(y) \leq b(x)$). Then, every shape is the sum of a finite set of disjoint segments.

Every segment can be identified with a carrier, indeed, indefinitely many carriers (although it is convenient to define carriers uniquely, especially for computer implementations.) In principle, carrier shapes possess structurally and topologically interesting properties for shapes. For example, a curve segment is carried by every curve in which that segment can be embedded, a surface segment is carried by every surface in which that segment can be embedded and so on. Two segments are *coequal* (i.e., collinear, coplanar, cohyperplanar, etc.) if there is a carrier that carries both shapes. Note that all carriers of a segment are coequal. This is obvious for rectilinear line of plane segments, and also for algebraic curves and surfaces. Examples of algebraic curves and surfaces are rational B-splines, NURBS, polynomial surfaces and Bezier surfaces. Most useful curves and surfaces are defined algebraically, or can be approximated as algebraic (Hoffmann, 1989).

Carriers extend to shapes; the carrier of a shape is the sum of the carriers of its segments. A shape is *coequal* if all its segments are coequal. A segment is necessarily coequal. In practice, carriers are specified by equations, in a unique way. The equation of a carrier can be used to define *coincidence* of a shape with a carrier if the former satisfies the equation of the latter. The carriers of segments act as a filter in our process of distinguishing categories of shapes.

In this model of shape algebra, pictorial equivalence does not imply shape equality. Figure 1 shows a curve segment s that is embedded in two different (curved) carriers given by equations $g(x, y) = 0$ and $h(x, y) = 0$. Although the curve segment s of type g is pictorially identical to the curve segment s of type h, the two are not the same shape. Pictorial equivalence assumes a pictorial scale and resolution at which equivalence is assessed. Equality between shapes requires that their carriers have identical equations and their boundaries are identical. In other words, the embedding character of a shape is integral to its identity. It is this embedding character of a shape rather than its point set equivalence that distinguishes this approach from conventional geometrical modeling. For linear shapes defined on a Euclidean geometry, this distinction between pictorial equivalence and shape equality does not arise since for any linear segment its carriers can be carried by a single largest carrier. The algebraic approach applies to linear as well as curved segments on condition that carriers are selected in such a way that pictorially identical segments always have the same carrier. This applies to all carriers that are algebraically defined, because two curves with algebraically defined carriers can never have more than a finite number of intersection points (within a finite area), and thus can never be pictorially identical (at the right level of detail).

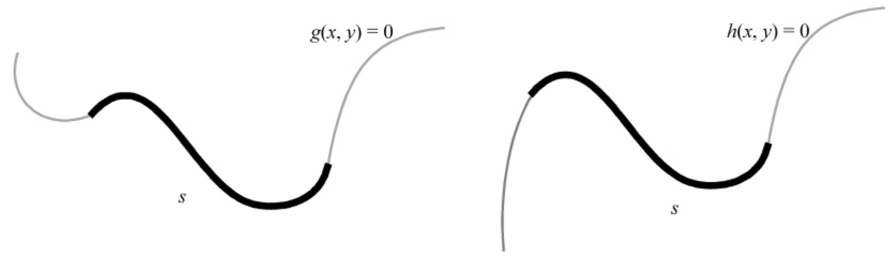

Figure 1. Two unequal curved segments that are pictorially the same.

Pictorial equivalence can only be assessed through approximation and it is generally considered only for a subset of potential carriers, e.g., NURBS. Pictorial equivalence can still be applied, however, outside of the algebraic model. Curves or surfaces can be approximated and compared and can be found approximately identical (or pictorially equivalent), but this identity is not considered within the algebraic model.

References

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